ISSN 2070-0466, p-Adic Numbers, Ultrametric Analysis and Applications, 2014, Vol. 6, No. 4, pp. 303–309. © Pleiades Publishing, Ltd., 2014.

RESEARCH ARTICLES

Recursion over Partitions*

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Received August 29, 2014

Abstract—The paper demonstrates how to apply a recursion on the fundamental concept of number. We propose a generalization of the partitions of a positive integer n, by defining new combinatorial objects, namely *sub-partitions*. A recursive formula is suggested, designated to solve the associated enumeration problem. It is highlighted that sub-partitions provide a good language to study rooted phylogenetic trees.

DOI: 10.1134/S2070046614040050

Key words: *partition, sub-partition, phylogenetic tree.*

1. INTRODUCTION

A partition of a number n is a way to present it as a sum of non negative integer numbers when the order has no significance. An example to a partition of 3 is 3=1+2. The partition function is the number of different partitions of a specific number n and it is written as p(n). The partition function is first mentioned in one of Leibniz's letters to J. Bernoulli (1674). He observed the first few values of p(n), i.e. 1,2,3,5,7,11 and he considered the possibility that p(n) is a prime number for all n>1. Since p(7)=15, Leibniz pose a different problem: Is there an infinite number of integers n for which p(n) are primes. Until today there is no definitive formula to p(n), but there is an asymptotic formula to p(n) which was discovered by Ramanujan and Hardy. In this paper we look at a tree with numbers written on its leaves. The degree of this tree is the sum of these numbers. These trees are known as phylogenetic trees. In order to answer the question how many phylogenetic trees are there of degree n, we can use the partitions of the number. The answer to the general question of the number of trees is a recursion over the partitions of n.

Another problem with a similar answer is the counting of the number of parentheses when there is no significance to the order. Let us look, for example, at parentheses of order 2 ( )( );(( )). When we look at parentheses with order 3 there are 5 possibilities ( )( )( );(( ))( );(( )( ));( )(( ));((( ))). The general number of possibilities is calculated by the Catalan numbers. But in the specific problem when the order is not important like in the problem of phylogenetic trees the two possibilities ( )(( ))=(( ))( ) are identical. Every parentheses form of order n is related to some partition of n. On the other hand, for every partition we can define a family of parentheses formed by using a process of inner recursion.

*The text was submitted by the authors in English.

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2. A ROOTED PHYLOGENETIC TREE

A *plane tree* T can be defined recursively as a finite set of vertices, with a single distinguished vertex r called the *root* of T, and the remaining vertices forming an ordered partition $(T_1, T_2, \ldots, T_m)$ of m disjoint non-empty sets, each of which is also a plane tree. We will draw plane trees with the root on the top level. The edges connecting the root of the tree to the roots of $T_1, T_2, \ldots, T_m$ will be drawn from left to right on the second level. For each vertex v, the vertices in the next lower level adjacent to v are called the *children* of v, and v is said to be their *parent*. Clearly, each vertex other than r has exactly one parent. A vertex of T is called a *leaf* if it has no children (by convention, we assume that the empty tree, formed by a single vertex, has no leaves), otherwise it is said to be an *internal* vertex. The *outdegree* of a vertex v it is number of its children, and denoted by $deg(v)$. It is well-known that the number of plane trees with n edges equals

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number (see [8, Exe. 6.19e]).

A *rooted phylogenetic tree* of $[n] = \{1, 2, \ldots, n\}$ is a rooted tree with the following property: *(i)* the leaves of the tree are labeled with the elements of the set $[n]$; *(ii)* the sequence of the leaves is non-increasing if we read the tree by its preorder that means that for any two children a, b of an internal vertex v such that a is on the left side of v, the sum of the label of the leaves in the subtree of a is greater or equal to the sum of the leaves in the subtree of b. Phylogenetic trees are an essential tool to describe evolutionary relationships.

A *partition* of a positive integer n is a way of writing n as a sum of non-increasing sequences of positive integers. A summand in a partition is also called a *part* and the number of partitions of n is given by the partition function $p(n)$. For instance, the partitions of 3 are given by 3, $2 + 1$ and $1 + 1 + 1$, which implies that $p(3) = 3$. Up to now there is no explicit formula for $p(n)$, for example, see [1] to find several properties of the function $p(n)$.

To keep track of all partitions, we write the larger number on the left-hand-side, so $3 + 1$ appears here, while $1 + 3$ (which corresponds to the same partition) does not appear. We also order the partitions by writing the partition having the larger number prior to the one having the smaller number, so that the partition $3 + 1$ appears before the partition $2 + 2$, and $2 + 2$ appears before $2 + 1 + 1$. The different partitions of a number n describe something that resembles indistinguishability of items, that is, if we talk about three similar items we might prefer to use the partition 3, while if we talk about two similar items and an extra one which is different from them, we might prefer to use the partition $2 + 1$.

In this paper, we extend the concept of partitions of n to find a new combinatorial structure, which we call *sub-partition*. This structure provides a good language to describe the rooted phylogenetic trees, as it is discussed in the next section. With the use of sub-partitions we enumerate rooted phylogenetic trees of $[n]$ and provide a recursive formula. Klein and Shadmi [5] (see also [4]) discussed a notion similar to sub-partitions.

3. SUB-PARTITIONS

If $n = k + m$ is a partition of n, we can consider a sub-partition using the fact that (for instance) $k = t + s$, namely using the partition of each of the summands. Here, we assume that this is different from simply taking the "other" partition $n = t + s + m$ (after a proper ordering of the elements), since we want the sub-partition to be some kind of process internal to the summand. The process is defined after we have already "finished" the partition, as if we are now sorting according to a different property. As our first example, we can see that the two partitions of the number 2, namely $\{2\}$ and $\{1 + 1\}$, where we keep using the set notations as in the previous section, cannot yield anything new if we try to perform a sub-partition, as the two counted items can either be identical (which corresponds to the partition $\{2\}$) or different (which corresponds to the partition $\{1 + 1\}$). There is nothing more to do in this case. For the number 2, there are two partitions, and furthermore, there are also just two sub-partitions, and these are identical to its partitions. However, we may look at this scenario differently: any attempt to have additional sub-partitions (further partitioning of $\{1 + 1\}$ or of $\{2\}$) will lead via recursion to an infinite number of such sub-partitions.

Considering a more interesting example, since $\{3\}$, $\{2+1\}$ and $\{1+1+1\}$ are the partitions of 3, we believe that now it does in fact make sense to look at second-level partitions. As before, it is clear that it is meaningless to perform sub-partitions on the partition $\{1+1+1\}$, and on the partition $\{3\}$, as this would lead again to an infinite number of sub-partitions, via recursion. Thus, using the number 3 to help us finding a proper definition, we see that recursions may be used, yet must be applied carefully. For instance, in that case, it only makes sense to perform a sub-partition on the element $\{2+1\}$ only. The number 2 has two different partitions: "partition-a" which is $\{2\}$ and "partition-b" which is $\{1+1\}$. The process of performing a sub-partition on the number 3 by using a partition of the number 2 will thus lead to splitting the partition $\{2+1\}$ into two sub-partitions: if we replace the summand 2 in the element $\{2+1\}$ by its "partition-a" we get $\{\{2\}+1\}$ and if we replace the summand 2 in the element $\{2+1\}$ by its "partition -b" we get $\{\{1+1\}+1\}$. As this results from a sub-partitioning of the original partition $\{2+1\}$, we consider the element $\{\{1+1\}+1\}$ to be different from the element $\{1+1+1\}$, for the reasoning explained above.

For the number 3, we noticed that the elements $\{3\}$ and $\{1+1+1\}$, obtained at the first level of partition, cannot be further partitioned and it only makes sense to consider $\{2+1\}$. After the second level, we thus obtain four sub-partitions for the number 3: $\{3\}$, $\{\{2\}+1\}$, $\{\{1+1\}+1\}$ and $\{1+1+1\}$. Note that after the first partition we had $\{3\}$, $\{2+1\}$ and $\{1+1+1\}$, therefore the notation we use here helps us to distinguish a partition such as $\{2+1\}$ from a sub-partition such as $\{\{2\}+1\}$. Since $\{2\}$ is a singlet, the process of sub-partitioning has ended.

In the general case, terms that look like $\{a_1+a_2+\cdots+a_k\}$ in the original partition are now (after that sub-partitioning) containing at least two summands, where each summand a_i is either the number 1, or it is replaced by $\{\}$, where any partition of a_i can be inserted in the parentheses. For the number 3, there is no need of a second recursion, but in general, there can be many levels of recursion. At the second level of recursion the same process applies, and the recursion ends only when any term k appears directly inside the parentheses. Indeed, notice that each step in the recursion corresponds to adding internal parentheses. By applying the general procedure, we can directly compute (see Table 1) the number of sub-partitions of n for some small values. We denote this number by sp_n:

Table 1. Number sub-partitions sp_n for $n = 1, 2, \ldots, 6$.

n	$P(n)$	sp_n
1	1	1
2	2	2
3	3	4
4	5	11
5	7	30
6	11	96

The above examples are useful to clarify the next formal definition:

Definition 3.1. *A* sub-partition π *of the integer* $n \geq 2$ *can be constructed recursively as follows:*

- *either* $\pi = n$,
- *or for a fixed partition* $\lambda = \lambda_1\lambda_2\cdots\lambda_k$ *of* n *with at least* $k \leq n$ *parts,* π *is given by the sequence*

$$SP_{\lambda_1} + \ldots + SP_{\lambda_k},$$

where either $SP_m = \{\pi'\}$ *with* π' *is any sub-partition of* m *with at least two summands, or* $SP_m = m$. *In this case,* π *is said to be a* λ-sub-partition *of* n. *We assume that the sub-partition of* 1 *is* 1.

In this context, $SP_{\lambda_1}, \ldots, SP_{\lambda_k}$ are called the 1-parts *of the sub-partition of π. We define the ℓ-*parts *of π to be the* 1*-parts of all the $(\ell-1)$-parts of π. A* part *of π is a k-part of π, where $k \geq 1$. We denote the set of sub-partitions of n by SP_n and its cardinality by sp_n, that is, $sp_n = |SP_n|$.*

There is a clear bijection between the sub-partitions of n and phylogenetic rooted trees of $[n]$; this sequence is in [7, A141268]. For $n = 2$, we have two sub-partitions: 2 or 11-sub-partition $1 + 1$. For $n = 3$, we have four sub-partitions: 3, or 21-sub-partitions $2 + 1$, $\{1 + 1\} + 1$, or 111-sub-partition $1 + 1 + 1$, as shown in the previous section. The sub-partition $\pi = \{1 + 1\} + 1$ has two parts $\{1 + 1\}$ and 1; where 1 and 1 are the parts of $\{1 + 1\}$, which gives that the maltiest of parts of π are $\{1 + 1\}, 1, 1, 1$.

At first we obtain a recurrence relation for the number of sub-partitions of n. To do so we need the following particular case.

Lemma 3.2. *Let $\lambda = \underbrace{kk\cdots k}_{s\ times}\lambda' = k^s\lambda'$ be any partition of $n > ks$ such that the largest element of λ' is at most $k - 1$. Then the number of λ-sub-partition a_λ of n is given by*

$$\binom{sp_k + s - 1}{s} a_{\lambda'},$$

where $a_{\lambda'}$ is the number of λ'-sub-partitions of $n - ks$.

Proof. From the recursive construction of the sub-partitions of n, we obtain that any λ-sub-partition can be written as $\{SP_1 + \cdots + SP_s + SP'\}$, where SP_j, $j = 1, 2, \ldots, s$, is a sub-partition of k and SP' is any λ'-sub-partition of $n - ks$. Since the sum $SP_1 + \cdots + SP_s$ has the same value for any order of the terms $SP_1, \ldots, SP_s$, we obtain that the number of possibilities to construct a sub-partition of the form $SP_1 + \cdots + SP_s$ such that SP_j, $1 \leq j \leq s$, is given by $\binom{sp_k+s-1}{s}$. Hence, by the definition of $a_{\lambda'}$, we get that the number of λ-sub-partition of n is given by $\binom{sp_k+s-1}{s} a_{\lambda'}$, as claimed. □

Theorem 3.3. *The number of sub-partitions of n, sp_n, satisfies the following recurrence relation*

$$sp_n = 1 + \sum \prod_{j=1}^{m} \binom{sp_{k_j} + s_j - 1}{s_j},$$

where the sum is over all partitions $\lambda = (k_1)^{s_1}(k_2)^{s_2}\cdots(k_m)^{s_m}$ of n such that $n - 1 \geq k_1 > k_2 > \cdots > k_m \geq 1$.

Proof. Let $\lambda = (k_1)^{s_1}(k_2)^{s_2}\cdots(k_m)^{s_m}$ be any partition of n such that $n - 1 \geq k_1 > k_2 > \cdots > k_m \geq 1$. By Lemma 3.2, we have that the number of λ-sub-partition of n is given by

$$\prod_{j=1}^{m} \binom{sp_{k_j} + s_j - 1}{s_j}.$$

Summing over all possibilities of λ, we obtain the equation in the statement. In the formula, 1 counts the sub-partition $\{n\}$. □

Applying the theorem for $n = 1, 2, \ldots, 8$, we obtain $sp_1 = 1$, $sp_2 = 2$, $sp_3 = 4$, $sp_4 = 11$, $sp_5 = 30$, $sp_6 = 96$, $sp_7 = 308$, $sp_8 = 1052$.

Definition 3.4. *A ℓ-*sub-partition *π of n is a sub-partition of n such that any part of π it is sub-partition of at most ℓ. We denote the number of ℓ-sub-partitions of n by $sp_{n,\ell}$.*

Clearly there exists only one 1-sub-partition of n, namely $1 + 1 + \cdots + 1$. The above theorem gives that

$$sp_{n,2} = \sum_{\lambda = 2^b 1^a \text{ partition of } n} (a + 1) = \binom{\lfloor n/2 \rfloor + 2}{2},$$

for all $n \geq 3$. For instance, there are six 2-sub-partitions of 4, namely $2+2$, $2+1+1$, $2+\{1+1\}$, $1+1+1+1$, $\{1+1\}+1+1$ and $\{1+1\}+\{1+1\}$. Indeed, the above theorem gives the following corollary.

Corollary 3.5. *For $n > \ell \geq 1$,*

$$sp_{n,\ell} = \sum_{\lambda=\ell^{s_\ell}\cdots 2^{s_2}1^{s_1} \text{ parition of } n} \prod_{j=1}^{\ell} \binom{sp_j + s_j - 1}{s_j}.$$

Definition 3.6. *A* strict sub-partition π *of* n *is a λ-sub-partition of n such that the partition λ has no equal parts. The number of strict sub-partitions of n by ssp_n.*

From the above theorem, we get the following result:

Corollary 3.7. *The number of strict sub-partitions of n is given by*

$$ssp_n = 1 + \sum \prod_{j=1}^{m} ssp_{k_j},$$

where the sum is over all partitions $\lambda = k_1 k_2 \cdots k_m$ of n such that $n-1 \geq k_1 > k_2 > \cdots > k_m \geq 1$.

Applying the corollary for $n = 1, 2, \ldots, 8$, we get $ssp_1 = 1$, $ssp_2 = 1$, $ssp_3 = 2$, $ssp_4 = 3$, $ssp_5 = 6$, $ssp_6 = 12$, $ssp_7 = 28$ and $ssp_8 = 65$.

4. FORMS OF PARENTHESES

Let us examine all the possibilities to write parentheses in a correct manner. We call a possibility that contains n pairs of parentheses, a possibility of degree n. The "degree" of a possibility K is the number of pairs of parentheses it contains. There is only one possibility of degree 1: ( ).

There are 2 possibilities of parentheses of degree 2: (( )) or ( )( ).

The number of possibilities of degree 3 is 5:

$$()()();\ ()(());\ (()());\ (())();\ ((()))$$

The sequence defined by $a_n =$ the number of possibilities of degree n is basically the Catalan Sequence, and a_n are known as Catalan Numbers [Catalan numbers A000108]. The Catalan Sequence can be computed in the following manner:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Now let us go one step further and distinguish between possibilities only by what they contain and by order. For instance, from now on,$()(()) = (())()$. Inspired from "Laws of Form" written by Spencer Brown [10], we shall call the possibilities whose order is insignificant, "forms". Now let us create a "sub-partition" definition that fits those forms. Note that these sub-partitions (or forms) are applicable and relevant, for instance in Biology or Computer Science, when counting the number of ways to arrange n membranes in space. The number of forms (not possibilities) of degree 3 is 4 and not 5 as before. The forms are :

$$()()(); ()(()); (()()); ((()))$$

We define (n) as the collection of all forms of parentheses that are wrapped with brackets and inside them there is a form of degree n.

For instance: (2)= { (( )( )), ((( )))}. Note that (0)={( )}, (1)={(( ))} and (3) = {(( )( )( )), (( )(( ))), ((( )( ))), (((( ))))}.

Table 2. Number of forms for $n = 4$.

Partition	The recursion	The forms that comes out of it
$1+1+1+1$	()()()()	1
$1+1+2$	()()(1)	1
$1+3$	()(2)	2
$2+2$	(1)(1)	1
4	(3)	4
Total		9

This definition will assist us in computing the number of forms of degree n using a recursive process on partitions of a number, by replacing the addend Kj in a partition, by(Kj -1) — that is, outer parentheses and inside them a form of degree Kj -1 .

Examples for n=4:

In order to compute the number of forms of degree n, we start from the set of all partitions of a number. Now we do a recursion on every addend but subtracted by 1 (because of the outer brackets).

Both sequences (forms and sub-partitions) can be computed using recursion on the partitions of n.We denote this number by pf_n.

The number of parentheses forms s of n, pf_n, satisfies the following recurrence relation.

$$pf_n = \sum \prod_{j=1}^{m} \binom{pf_{k_j-1} + s_j - 1}{s_j},$$

where the sum is over all partitions $\lambda = (k_1)^{s_1}(k_2)^{s_2} \cdots (k_m)^{s_m}$ of n such that $n-1 \geq k_1 > k_2 > \cdots > k_m \geq 1$.

Table 3. An example of the number of partitions, sub-partitions and paratheses forms for some small integers.

n	1	2	3	4	5	6
$p(n)$	1	2	3	5	7	11
$sp(n)$	1	2	4	11	30	96
$pf(n)$	1	2	4	9	20	48

5. CONCLUSION

Recently Ken Ono published a breakthrough in the theory of partitions. He discovered a fractal nature in the structure of divisibility mod a prime number p which is an extension to the known results of Ramanujan. In our paper we present a simpler use of fractals in the theory of partitions.

In the process of creating mathematics we invent new symbols and enrich their meaning by using recursion. In this paper we have demonstrated how to apply a recursion on the fundamental concept of number. We believe that it might be possible to generate more new structures which are based on this idea.

We mention briefly some applications of partitions of natural numbers. In [4] applications of partition theory to quantum physics and quantum information theory were considered. Partitions can be used for study of the fine structure of spectra — splitting of degenerate electron levels under the action of external electromagnetic field. Another possible application might be to give a description to the pilot-wave theory by Louis de Broglie as supported by the experiment on fluid mechanics by John Bush. We can also mention applications to non-Archimedean physics. The idea that the Archimedean axiom can be violated on the Planck distance was debated a lot, especially in superstring theory and cosmology, see, e.g. [11, 12]. The basic mathematical objects of non-Archimedean theoretical physics are ultrametric spaces. Geometrically such spaces are represented by trees; including homogeneous p-adic trees. Partition theory provides a possibility for number-theoretic representation of trees and hence ultrametric space; hence, it establishes a deeper connection between the non-Archimedean physics and number theory.

ACKNOWLEDGMENTS

Moshe Klein wants to thank Doron Shadmi for introducing to him the notion of Organic numbers and for the common work during the years.

M. K. also thanks Tal Mor for helping him to bring the ideas to rigorous mathematical presentation and to Toufik Mansour and Simone Severini for their important contribution to this paper including the idea of rooted phylogenetic trees.

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